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A New Perspective on Willems' Fundamental Lemma

Universality of Persistently Exciting Inputs

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Introduction

- The fundamental lemma was initially proven in (Willems et al., 2005).

A note on persistency of excitation

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Abstract

We prove that if a component of the response signal of a controllable linear time-invariant system is persistently exciting of sufficiently high order, then the windows of the signal span the full system behavior. This is then applied to obtain conditions under which the state trajectory of a state representation spans the whole state space. The related question of when the matrix formed from a state sequence has linearly independent rows from the matrix formed from an input sequence and a finite number of its shifts is of central importance in subspace system identification.

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    - design of stabilizing feedback gains (De Persis and Tesi, 2019), and
    - construction of predictive controllers (Coulson et al., 2019), (Berberich et al., 2020).
- Extensions:
  - multiple datasets (van Waarde et al., 2020),
  - parameter-varying systems (Verhoek et al., 2021),
  - robust version (Coulson et al., 2022),
  - frequency domain counterpart (Meijer et al., 2023),
  - continuous-time systems (Rapisarda et al., 2023),
  - stochastic systems (Pan et al., 2022),
  - descriptor systems (Schmitz et al., 2022),
  - switched systems (Petreczky and Bako, 2023),
  - 2D systems (Rapisarda and Zhang, 2024),
  - and several classes of nonlinear systems.

Problem Statement

Let $n, m, p \in \mathbb{N}$. Consider the input-state-output system

$$\begin{aligned} x(t+1) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t) + Du(t). \end{aligned} \tag{$\star$}$$

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We identify system $(\star)$ with the quadruple $(A, B, C, D) \in \mathcal{M}$, where

$$\mathcal{M} := \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m} \times \mathbb{R}^{p\times n} \times \mathbb{R}^{p\times m}.$$

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For an $(A, B, C, D) \in \mathcal{M}$, we denote its input-output behavior by

$$\mathfrak{B}(A, B, C, D) := \{(u, y) : \mathbb{Z}_+ \to \mathbb{R}^m \times \mathbb{R}^p \mid \exists x : \mathbb{Z}_+ \to \mathbb{R}^n \text{ such that } (\star) \text{ holds}\},$$

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Given $u : \mathbb{Z}_+ \to \mathbb{R}^n$ and $T \in \mathbb{N}$, we define $u_{[0,T-1]} := \begin{bmatrix} u(0)^\top & \cdots & u(T-1)^\top \end{bmatrix}^\top$.

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For an $(A, B, C, D) \in \mathcal{M}$, we denote its k-restricted input-output behavior by

$$\mathfrak{B}_k(A, B, C, D) := \left\{ \begin{bmatrix} u_{[0,k-1]} \\ y_{[0,k-1]} \end{bmatrix} \mid (u, y) \in \mathfrak{B}(A, B, C, D) \right\},$$

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and its k-restricted input-state behavior by $\mathfrak{B}_k(A, B) := \mathfrak{B}_k(A, B, I_n, 0)$.

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- The Hankel matrix of $u_{[0,T-1]}$ of depth k is denoted by

$$\mathcal{H}_k(u_{[0,T-1]}) := \begin{bmatrix} u(0) & u(1) & \cdots & u(T-k) \\ u(1) & u(2) & \cdots & u(T-k+1) \\ \vdots & \vdots & \ddots & \vdots \\ u(k-1) & u(k) & \cdots & u(T-1) \end{bmatrix}.$$

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- We say $u_{[0,T-1]}$ is PE of order k if $\mathcal{H}_k(u_{[0,T-1]})$ has full row rank.

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Willems et al.'s fundamental lemma

Let $T \in \mathbb{N}$, $L \in [1, T]$, and $(A, B, C, D) \in \mathcal{M}_{\text{cont}}$. If $u_{[0,T-1]}$ is PE of order $n + L$, then

$$\operatorname{rank} \begin{bmatrix} \mathcal{H}_L(u_{[0,T-1]}) \\ \mathcal{H}_1(x_{[0,T-L]}) \end{bmatrix} = n + Lm$$

for all $x_{[0,T-L]}$ satisfying

$$\begin{bmatrix} u_{[0,T-L]} \\ x_{[0,T-L]} \end{bmatrix} \in \mathfrak{B}_{T-L+1}(A, B).$$

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- PE of order $n + L$ requires a sufficiently long trajectory, namely

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Let $x(0) = \begin{bmatrix} x_1(0) & x_2(0) \end{bmatrix}^\top$, $T = 3$, $L = 1$. Let $u(0) = 1$, and $u(1) = u(2) = 0$.

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We have $\begin{bmatrix} \mathcal{H}_1(u_{[0,2]}) \\ \mathcal{H}_1(y_{[0,2]}) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ x_1(0) & x_2(0) & 1 \end{bmatrix}$, which has full row rank for all $x(0) \in \mathbb{R}^2$.

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Definition

Let $T \in \mathbb{N}$ and $L \in [1, T]$. An input $u_{[0,T-1]}$ is called universal for determining the L-restricted behavior if for every $(A, B, C, D) \in \mathcal{M}_{\text{cont}}$ and every $y_{[0,T-1]}$ satisfying $\begin{bmatrix} u_{[0,T-1]} \\ y_{[0,T-1]} \end{bmatrix} \in \mathfrak{B}_T(A, B, C, D)$ we have

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- What are necessary and sufficient conditions for an input to be universal?

Main Results

Theorem

Let $T \in \mathbb{N}$ and $L \in [1, T]$. An input $u_{[0,T-1]}$ is universal for determining the L-restricted behavior if and only if it is persistently exciting of order $n + L$.

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- The "if" part follows from Willems et al.'s fundamental lemma.
- For the "only if" part, given an input that is not PE, we show how to construct
  - a controllable system and
  - an initial condition

  such that the resulting input-output data is rank deficient.

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- Let $\eta_0, \ldots, \eta_{n+L-1} \in \mathbb{R}^m$ be such that

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- Take $A \in \mathbb{R}^{n\times n}$ and $\zeta \in \mathbb{R}^n$ such that
  - (A, ζ) is controllable,
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$$E_{i-1} = AE_i + \zeta \eta_i^\top.$$

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- Take $B = E_{-1}$ and $x(0) = -\sum_{i=0}^{n+L-2} E_i u(i)$.
- The resulting trajectory satisfies rank $\begin{bmatrix} \mathcal{H}_L(u_{[0,T-1]}) \\ \mathcal{H}_1(x_{[0,T-L]}) \end{bmatrix} < n + Lm$.

Example 1

- Let $n = 3$, $m = 2$, $L = 1$, and $T = 8$. Consider the input signal

t	0	1	2	3	4	5	6	7
$u(t)$	-0.46 1.86	-1.09 -0.87	0.9 -1.42	1.38 1.06	1.24 1.6	1.54 -1.94	0.22 1.11	-1.68 -1.03

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- Compute $\begin{bmatrix}\eta_0^\top & \eta_1^\top & \eta_2^\top & \eta_3^\top\end{bmatrix} \in \operatorname{leftker} \mathcal{H}_4(u_{[0,7]}) \backslash \{0\}$ as

$$\begin{bmatrix}\eta_0 & \eta_1 & \eta_2 & \eta_3\end{bmatrix} = \begin{bmatrix} -0.2039 & -0.1162 & 0.0598 & -0.0782 \\ 0.4924 & 0.4964 & 0.6281 & 0.2277 \end{bmatrix}.$$

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- Choose A and ζ, with the entries drawn uniformly at random:

$$A = \begin{bmatrix} -0.2675 & -1.3834 & -1.4581 \\ -0.2088 & -2.2509 & -2.5228 \\ 2.3349 & -1.5816 & 1.0206 \end{bmatrix}, \quad \zeta = \begin{bmatrix} -0.4351 \\ 0.0482 \\ 0.8496 \end{bmatrix}.$$

The pair (A, ζ) is controllable.

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- Compute E_i, $i \in [-1, 2]$, from the recursion:

$$E_2 = \begin{bmatrix} 0.0340 & -0.0991 \\ -0.0038 & 0.0110 \\ -0.0664 & 0.1935 \end{bmatrix}, \quad E_1 = \begin{bmatrix} 0.0669 & -0.5441 \\ 0.1718 & -0.4618 \\ 0.0684 & 0.4823 \end{bmatrix},$$

$$E_0 = \begin{bmatrix} -0.3047 & -0.1349 \\ -0.5788 & -0.0398 \\ -0.1444 & 0.3741 \end{bmatrix}, \quad E_{-1} = \begin{bmatrix} 1.1815 & -0.6687 \\ 1.7209 & -0.8023 \\ -0.1166 & 0.5480 \end{bmatrix}.$$

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- Take $B = E_{-1}$ and compute $x(0) = -\sum_{i=0}^{2} E_i u(i)$ to have

$$B = \begin{bmatrix} 1.1815 & -0.6687 \\ 1.7209 & -0.8023 \\ -0.1166 & 0.5480 \end{bmatrix}, \quad x(0) = \begin{bmatrix} -0.461 \\ -0.3879 \\ 0.0665 \end{bmatrix}.$$

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- The state trajectory generated by (A, B) starting from $x(0)$:

t	0	1	2	3	4	5	6	7
	-0.4610	-1.2243	0.6834	1.1065	-0.4624	0.7518	0.4177	1.0417
$x(t)$	-0.3879	-1.4824	0.7041	1.3940	-0.3905	1.2941	-0.0876	2.2183
	0.0665	0.6780	-0.1718	-0.5763	0.2108	0.4853	-1.0387	0.6362

which satisfies $\operatorname{rank} \begin{bmatrix} \mathcal{H}_1(u_{[0,7]}) \\ \mathcal{H}_1(x_{[0,7]}) \end{bmatrix} < 5$.

Example 2

- Consider a single-state multi-input system with $n = 1$ and $m = 2$, as

$$x(t+1) = ax(t) + \begin{bmatrix} b_1 & b_2 \end{bmatrix} u(t).$$

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- Let $T = 7$, $L = 2$, and the input as

t	0	1	2	3
$u(t)$	-1.24 0.67	0.35 0.7	-0.56 0.48	1.24 -1.92

t	4	5	6
$u(t)$	-1.66 1.9	0.61 -1.08	-0.39 -1.51

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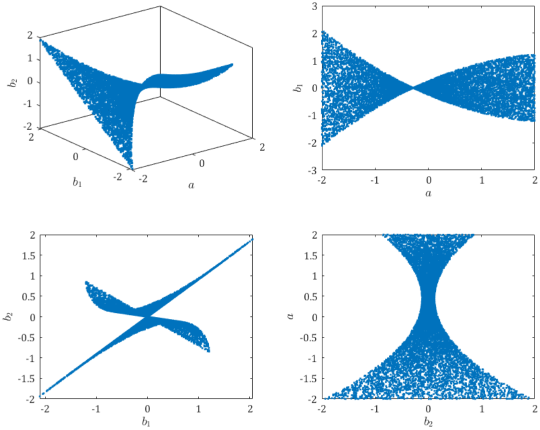

Figure: A set of controllable systems that generate rank-deficient data.

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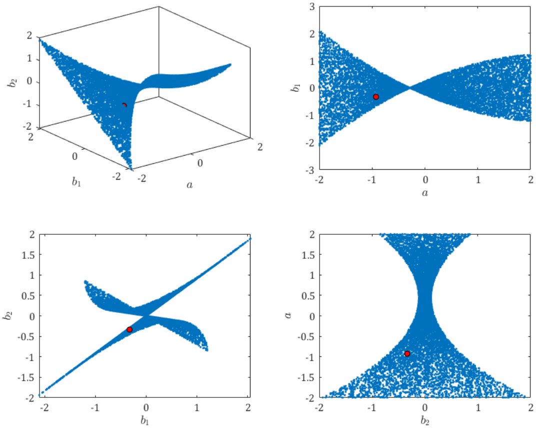
- In particular, for the red dot:

$$a = -0.9262,$$
$$b_1 = -0.3273,$$
$$b_2 = -0.3356.$$



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- ▷ The trajectory generated by this system, starting from $x(0) = 0.5561$, is such that

$$\operatorname{rank} \begin{bmatrix} \mathcal{H}_2(u_{[0,6]}) \\ \mathcal{H}_1(x_{[0,5]}) \end{bmatrix} = 4$$

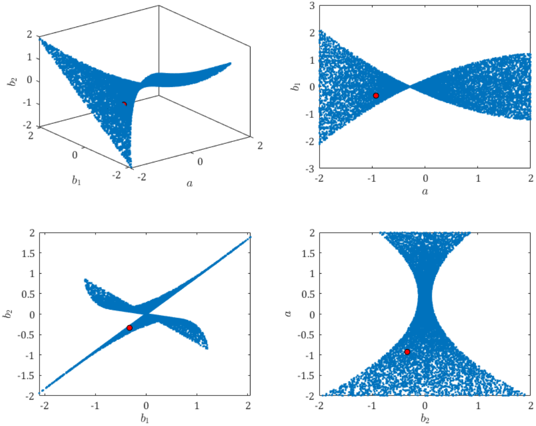

Figure: A set of controllable systems that generate rank-deficient data.

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